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Some generating functions for the Jacobi polynomials

by

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1. Introduction

In a series of recent papers [8] through [11], Srivastava has developed a number of linear, bilinear and bilateral generating functions for the Jacobi polynomials $\{P_n^{(\alpha,\beta)}(x)\,|\,n=0, 1, 2, \cdots\}$ defined by (see, for instance, [15], p. 68)

$$(1.1)\qquad P_n^{(\alpha,\beta)}(x)=\sum_{k=0}^{n}\binom{n+\alpha}{k}\binom{n+\beta}{n-k}\left(\frac{x-1}{2}\right)^{n-k}\left(\frac{x+1}{2}\right)^{k}.$$

The object of the present note is to derive another class of generating functions for these polynomials. Our main result, which admits itself of a further generalization discussed in §4 and which will subsequently be shown to incorporate as its particular cases a large number of generating functions obtained by earlier writers, is given by

$$(1.2)\qquad \sum_{n=0}^{\infty}\frac{\prod_{j=1}^{\nu}(\alpha_j)_{n\lambda_j}}{(\alpha+1)_n(\beta+1)_n\prod_{j=1}^{\sigma}(\beta_j)_{n\mu_j}}P_n^{(\alpha,\beta)}(x)F\begin{matrix}\nu\colon A';\cdots;A^{(s)}\\ \sigma\colon B';\cdots;B^{(s)}\end{matrix}$$

$$\left(\begin{matrix}[(\alpha_\nu+n\lambda_\nu)\colon\theta',\cdots,\theta^{(s)}]\colon[(a')\colon\phi'];\cdots;[(a^{(s)})\colon\phi^{(s)}];\\ [(\beta_\sigma+n\mu_\sigma)\colon\psi',\cdots,\psi^{(s)}]\colon[(b')\colon\delta'];\cdots;[(b^{(s)})\colon\delta^{(s)}];\end{matrix}\, z_1,\cdots,z_s\right)t^n$$

$$=F\begin{matrix}\nu\colon A';\cdots;A^{(s)};0;0\\ \sigma\colon B';\cdots;B^{(s)};1;1\end{matrix}\left(\begin{matrix}[(\alpha_\nu)\colon\theta',\cdots,\theta^{(s)},\lambda,\lambda]\colon[(a')\colon\phi'];\cdots;\\ [(\beta_\sigma)\colon\psi',\cdots,\psi^{(s)},\mu,\mu]\colon[(b')\colon\delta'];\cdots;\end{matrix}\right.$$

$$\left.\begin{matrix}[(a^{(s)})\colon\phi^{(s)}];\quad \text{——};\quad \text{——};\\ [(b^{(s)})\colon\delta^{(s)}];[\alpha+1\colon 1];[\beta+1\colon 1];\end{matrix}\, z_1,\cdots,z_s,\tfrac{1}{2}(x-1)t,\tfrac{1}{2}(x+1)t\right),$$

where λ_i, $\theta_i^{(j)}$, $i=1,\cdots,\nu$ and $j=1,\cdots,s$; μ_i, $\psi_i^{(j)}$, $i=1,\cdots,\sigma$ and $j=1,\cdots,s$; $\phi_i^{(j)}$, $\delta_k^{(j)}$, $i=1,\cdots,A^{(j)}$, $j=1,\cdots,s$ and $k=1,\cdots,B^{(j)}$ are all real and positive, (α_ν) is taken to abbreviate the sequence of ν parameters $\alpha_1,\cdots,\alpha_\nu$, $(a^{(j)})$ stands for the sequence of $A^{(j)}$ parameters

$$a_1^{(j)},\cdots,a_{A^{(j)}}^{(j)},\qquad j=1,\cdots,s,$$

with similar interpretations for (β_σ) and $(b^{(j)})$, $j=1,\cdots,$ s, and $F(z_1,\cdots,z_s)$

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denotes the generalized Lauricella function of s complex variables $z_1, \cdots, z_s$ which we introduced earlier (see [13], p. 454 for details).

2. Proof of the formula (1.2)

For convenience, let $\sum m_i\varepsilon_i^{(j)}$ abbreviate the s-term sum

$$\sum_{i=1}^{s} m_i\varepsilon_i^{(j)} = m_1\varepsilon_1^{(j)} + \cdots + m_s\varepsilon_s^{(j)}\ , \tag{2.1}$$

for every j as indicated. Also let Ω denote the first member of the formula (1.2).

Then, by (1.1) and the definition of the generalized Lauricella function (cf. [13], p. 454, eq. (4.1) et seq.), we have

$$\Omega = \sum_{n,m_1,\cdots,m_s=0}^{\infty} \frac{\prod_{j=1}^{\nu}(\alpha_j)_{n\lambda_j+\Sigma m_i\theta_i^{(j)}} \prod_{j=1}^{A'}(a'_j)_{m_1\phi'_j} \cdots \prod_{j=1}^{A^{(s)}}(a_j^{(s)})_{m_s\phi_j^{(s)}}}{\prod_{j=1}^{\sigma}(\beta_j)_{n\mu_j+\Sigma m_i\psi_i^{(j)}} \prod_{j=1}^{B'}(b'_j)_{m_1\delta'_j} \cdots \prod_{j=1}^{B^{(s)}}(b_j^{(s)})_{m_s\delta_j^{(s)}}} \cdot \frac{z_1^{m_1}}{m_1!} \cdots \frac{z_s^{m_s}}{m_s!} \sum_{k=0}^{n} \frac{[\frac{1}{2}(x-1)t]^{n-k}[\frac{1}{2}(x+1)t]^k}{(\alpha+1)_{n-k}(\beta+1)_k(n-k)!\,k!}$$

$$= \sum_{n,k,m_1,\cdots,m_s=0}^{\infty} \frac{\prod_{j=1}^{\nu}(\alpha_j)_{n\lambda_j+k\lambda_j+\Sigma m_i\theta_i^{(j)}} \prod_{j=1}^{A'}(a'_j)_{m_1\phi'_j} \cdots \prod_{j=1}^{A^{(s)}}(a_j^{(s)})_{m_s\phi_j^{(s)}}}{\prod_{j=1}^{\sigma}(\beta_j)_{n\mu_j+k\mu_j+\Sigma m_i\psi_i^{(j)}} \prod_{j=1}^{B'}(b'_j)_{m_1\delta'_j} \cdots \prod_{j=1}^{B^{(s)}}(b_j^{(s)})_{m_s\delta_j^{(s)}}} \cdot \frac{1}{(\alpha+1)_n(\beta+1)_k} \frac{z_1^{m_1}}{m_1!} \cdots \frac{z_s^{m_s}}{m_s!} \frac{[\frac{1}{2}(x-1)t]^n}{n!} \frac{[\frac{1}{2}(x+1)t]^k}{k!}\ , \tag{2.2}$$

whence the formula (1.2) follows on interpreting this last multiple series (2.2) as the generalized Lauricella function of $s+2$ variables

$$z_1, \cdots, z_s, \tfrac{1}{2}(x-1)t, \tfrac{1}{2}(x+1)t\ .$$

3. Particular cases

The most interesting special cases of our generating function (1.2) would seem to occur when the generalized $F(z_1, \cdots, z_s)$-function is reduced to one or the other of Lauricella's four functions of s variables, viz. $F_A^{(s)}$, $F_B^{(s)}$, $F_C^{(s)}$ and $F_D^{(s)}$ (see [5], p. 113). Indeed it is known (cf. [13], pp. 454–455) that if the positive constants θ's, ϕ's, ψ's and δ's are all chosen as unity, then

$$F\begin{matrix}1:1;\cdots;1\\0:1;\cdots;1\end{matrix}(z_1, \cdots, z_s) \tag{3.1}$$

will correspond to Lauricella's $F_A^{(s)}$,

$$F\begin{matrix}0:2;\cdots;2\\1:0;\cdots;0\end{matrix}(z_1, \cdots, z_s) \tag{3.2}$$

to the $F_B^{(s)}$-function,

$$F\begin{matrix} 2:0;\cdots;0 \\ 0:1;\cdots;1 \end{matrix}(z_1, \cdots, z_s) \tag{3.3}$$

to $F_C^{(s)}$, and

$$F\begin{matrix} 1:1;\cdots;1 \\ 1:0;\cdots;0 \end{matrix}(z_1, \cdots, z_s) \tag{3.4}$$

to the fourth function $F_D^{(s)}$ of Lauricella ([5], p. 113).

For instance, by specializing the various parameters in (1.2) to suit case (3.3) above and letting $\alpha_1=\gamma$, $\alpha_2=\delta$, $\lambda_1=\lambda_2=1$, we obtain the generating function

$$\begin{aligned}(3.5)\quad &\sum_{n=0}^{\infty}\frac{(\gamma)_n(\delta)_n}{(\alpha+1)_n(\beta+1)_n}P_n^{(\alpha,\beta)}(x)\\ &\quad\cdot F_C^{(s)}[\gamma+n,\ \delta+n;\ \rho_1,\cdots,\rho_s;\ z_1,\cdots,z_s]t^n\\ &=F_C^{(s+2)}[\gamma,\ \delta;\ \rho_1,\cdots,\rho_s,\ \alpha+1,\ \beta+1;\ z_1,\cdots,z_s,\ \tfrac{1}{2}(x-1)t,\ \tfrac{1}{2}(x+1)t]\ ,\end{aligned}$$

which essentially is the formula (3), p. 345 of Saxena [7]. Note that for $z_1=(y-1)/(y+1)$ and $z_i=0$, $2\leqq i\leqq s$, formula (3.5) would reduce to a recent generalization ([6], p. 430, eq. (4)) of the late Professor Bailey's well-known bilinear generating function for the Jacobi polynomials (see [2], p. 9, eq. (2.1)).

Next we consider the special case of our formula (1.2) when $z_1=\cdots=z_s=0$. In terms of the generalized Kampé de Fériet function, defined and studied earlier by us (cf. [12], p. 199; see also [13], p. 450), we thus obtain

$$\begin{aligned}(3.6)\quad &\sum_{n=0}^{\infty}\frac{\prod_{j=1}^{\nu}\Gamma(\alpha_j+n\lambda_j)}{\Gamma(\alpha+n+1)\Gamma(\beta+n+1)\prod_{j=1}^{\sigma}\Gamma(\beta_j+n\mu_j)}P_n^{(\alpha,\beta)}(x)t^n\\ &=S\begin{matrix}\nu:0;0\\ \sigma:1;1\end{matrix}\left(\begin{matrix}[(\alpha_\nu):\lambda,\ \lambda]: & \text{——}; & \text{——};\\ [(\beta_\sigma):\mu,\ \mu]: & [\alpha+1:1]; & [\beta+1:1];\end{matrix}\ \tfrac{1}{2}(x-1)t,\ \tfrac{1}{2}(x+1)t\right).\end{aligned}$$

In particular, if in the last formula (3.6) we replace ν by $\nu+2$, let $\alpha_{\nu+1}=\alpha+1$, $\alpha_{\nu+2}=\beta+1$, and choose $\lambda_i=1=\mu_j$, where $i=1,\cdots,\nu+2$ and $j=1,\cdots,\sigma$, then we have the elegant generating function

$$\begin{aligned}(3.7)\quad &\sum_{n=0}^{\infty}\frac{\prod_{j=1}^{\nu}(\alpha_j)_n}{\prod_{j=1}^{\sigma}(\beta_j)_n}P_n^{(\alpha,\beta)}(x)t^n\\ &=F\left[\begin{matrix}\alpha+1,\ \beta+1,\ (\alpha_\nu): & \text{——}; & \text{——};\\ (\beta_\sigma): & \alpha+1; & \beta+1;\end{matrix}\ \tfrac{1}{2}(x-1)t,\ \tfrac{1}{2}(x+1)t\right],\end{aligned}$$

where $F[\xi, \eta]$ denotes Kampé de Fériet's double hypergeometric function ([1], p. 150) in the contracted notation of Burchnall and Chaundy ([4], p. 112).

Incidentally, formula (3.7) was derived earlier by us ([14], eq. (5.8)) as a special case of a summation formula associated with generalized hypergeometric functions. Note also that if $\nu=\sigma=2$, $\alpha_1=\gamma$, $\alpha_2=\delta$, $\beta_1=\alpha+1$, $\beta_2=\beta+1$, then (3.7) will reduce at once to Brafman's generating function ([3], p. 943)

$$\tag{3.8} \begin{aligned}&\sum_{n=0}^{\infty} \frac{(\gamma)_n(\delta)_n}{(\alpha+1)_n(\beta+1)_n} P_n^{(\alpha,\beta)}(x)t^n \\ &\quad = F_4[\gamma, \delta; \alpha+1, \beta+1; \tfrac{1}{2}(x-1)t, \tfrac{1}{2}(x+1)t] ,\end{aligned}$$

where F_4 denotes Appell's fourth function defined by ([1], p. 14)

$$\tag{3.9} F_4[\alpha, \beta; \gamma, \gamma'; x, y] = \sum_{m,n=0}^{\infty} \frac{(\alpha)_{m+n}(\beta)_{m+n}}{(\gamma)_m(\gamma')_n} \frac{x^m}{m!} \frac{y^n}{n!} .$$

In conclusion, it may be of interest to remark that since

$$\tag{3.10} P_n^{(\alpha,\alpha)}(x) = \binom{n+\alpha}{n}\binom{n+2\alpha}{n}^{-1} C_n^{\alpha+1/2}(x)$$

and

$$\tag{3.11} P_n^{(0,0)}(x) = C_n^{1/2}(x) = P_n(x) ,$$

where $C_n^{\alpha}(x)$ denotes the Gegenbauer (or ultraspherical) polynomial, and $P_n(x)$ is the Legendre polynomial of degree n in x, our formula (1.2) and its aforementioned consequences can be appropriately specialized to deduce generating functions for ultraspherical and Legendre polynomials. Also, since ([15], p. 103)

$$\tag{3.12} \lim_{\beta\to\infty} P_n^{(\alpha,\beta)}\left(1-\frac{2x}{\beta}\right) = L_n^{(\alpha)}(x) ,$$

where $L_n^{(\alpha)}(x)$ is the Laguerre polynomial of order α and degree n in x, suitable limiting cases of results obtained here will lead us to generating functions for Laguerre polynomials.

4. A further generalization

A closer examination of the proof of formula (1.2) would suggest the existence of its straightforward generalization in which the Jacobi polynomial is replaced by the finite sum

$$\tag{4.1} S_n(x) = \sum_{k=0}^{n} \frac{\prod_{j=1}^{p} (a_j)_{n\xi_j - k\xi_j} \prod_{j=1}^{m} (c_j)_{k\eta_j}}{\prod_{j=1}^{q} (b_j)_{n\zeta_j - k\zeta_j} \prod_{j=1}^{r} (d_j)_{k\varepsilon_j}} \frac{[\frac{1}{2}(x-1)]^{n-k}}{(n-k)!} \frac{[\frac{1}{2}(x+1)]^k}{k!} ,$$

where ξ_i, $i=1, \cdots, p$; ζ_j, $j=1, \cdots, q$; η_h, $h=1, \cdots, m$; ε_l, $l=1, \cdots, r$ are real and positive. Thus if, for convenience, we denote the generalized Lauricella function on the left-hand side of (1.2) by

$$F\begin{matrix}\nu: A'; \cdots; A^{(s)} \\ \sigma: B'; \cdots; B^{(s)}\end{matrix}(z_1, \cdots, z_s) ,$$

then the formula obtained is

$$\sum_{n=0}^{\infty} \frac{\prod_{j=1}^{\nu} (\alpha_j)_{n\lambda_j}}{\prod_{j=1}^{\sigma} (\beta_j)_{n\mu_j}} S_n(x) F\begin{matrix}\nu: A'; \cdots; A^{(s)} \\ \sigma: B'; \cdots; B^{(s)}\end{matrix}(z_1, \cdots, z_s) t^n \tag{4.2}$$
$$= F\begin{matrix}\nu: A'; \cdots; A^{(s)}; p; m \\ \sigma: B'; \cdots; B^{(s)}; q; r\end{matrix}\left(\begin{matrix}[(\alpha_\nu): \theta', \cdots, \theta^{(s)}, \lambda, \lambda]: [(a'): \phi']; \cdots; \\ [(\beta_\sigma): \psi', \cdots, \psi^{(s)}, \mu, \mu]: [(b'): \delta']; \cdots;\end{matrix}\right.$$
$$\left.\begin{matrix}[(a^{(s)}): \phi^{(s)}]; [(a_p): \xi]; [(c_m): \eta]; \\ [(b^{(s)}): \delta^{(s)}]; [(b_q): \zeta]; [(d_r): \varepsilon];\end{matrix} z_1, \cdots, z_s, \tfrac{1}{2}(x-1)t, \tfrac{1}{2}(x+1)t\right) ,$$

which can be shown to incorporate, as its particular cases, a large number of bilateral generating functions, known as well as new.

In particular, when the positive constants ξ's, η's, ζ's and ε's are all chosen as unity, the finite sum in (4.1) reduces to a generalized hypergeometric polynomial of degree n in x, and we have

$$S_n(x) = \frac{\prod_{j=1}^{p} (a_j)_n}{\prod_{j=1}^{q} (b_j)_n} \frac{[\frac{1}{2}(x-1)]^n}{n!} \tag{4.3}$$
$$\cdot {}_{q+m+1}F_{p+r}\left[\begin{matrix}-n, 1-(b_q)-n, (c_m); \\ 1-(a_p)-n, (d_r);\end{matrix} (-)^{p-q+1}\left(\frac{x+1}{x-1}\right)\right] .$$

Since

$$L_n^{(\alpha)}(z) = \frac{(\alpha+1)_n}{n!} {}_1F_1\left[\begin{matrix}-n; \\ \alpha+1;\end{matrix} z\right] = \frac{(-z)^n}{n!} {}_2F_0\left[\begin{matrix}-n, -\alpha-n; \\ \text{——};\end{matrix} -\frac{1}{z}\right] , \tag{4.4}$$

the right-hand side of (4.3) would correspond to

$$\frac{[\frac{1}{2}(x-1)]^n}{(\alpha+1)_n} L_n^{(\alpha)}\left(\frac{1+x}{1-x}\right)$$

when $p=q=m=r-1=0$, $d_1=\alpha+1$; or to

$$\frac{[\frac{1}{2}(x+1)]^n}{(\alpha+1)_n} L_n^{(\alpha)}\left(\frac{1-x}{1+x}\right)$$

when $p=q-1=m=r=0$, $b_1=\alpha+1$. Also, if $p-1=q=m=r=0$ and $a_1=-\alpha$, it gives us

$$\left(\frac{1-x}{2}\right)^n L_n^{(\alpha-n)}\left(\frac{x+1}{x-1}\right) .$$

Evidently, therefore, these special cases of formula (4.2) would yield bilateral generating functions for the Laguerre polynomials $L_n^{(\alpha)}(x)$ or $L_n^{(\alpha-n)}(x)$.

On the other hand, for $p=q-1=m=r-1=0$, $b_1=\alpha+1$, $d_1=\beta+1$, the $S_n(x)$ in (4.3) will reduce to

$$\frac{1}{(\alpha+1)_n(\beta+1)_n} P_n^{(\alpha,\beta)}(x) ,$$

and we arrive at our formula (1.2).

Next we recall the definition (1.1) in the form

$$P_n^{(\alpha,\beta)}(x)=\frac{(\beta+1)_n}{n!}\left(\frac{x-1}{2}\right)^n {}_2F_1\left[\begin{matrix}-n, \ -\alpha-n; \\ \beta+1;\end{matrix}\ \frac{x+1}{x-1}\right], \tag{4.5}$$

which would enable us to express the Jacobi polynomials $P_n^{(\alpha-n,\beta)}(x)$, $P_n^{(\alpha,\beta-n)}(x)$ and $P_n^{(\alpha-n,\beta-n)}(x)$ as particular cases of the $S_n(x)$ defined by (4.3). We are thus led to a number of bilateral generating functions for these special Jacobi polynomials. To quote one such formula derivable from (4.2), we have

$$\sum_{n=0}^{\infty}\frac{\prod_{j=1}^{\nu}(\alpha_j)_{n\lambda_j}}{\prod_{j=1}^{\sigma}(\beta_j)_{n\mu_j}} P_n^{(\alpha-n,\beta-n)}(x) F_{\sigma:\, B';\, \cdots;\, B^{(s)}}^{\nu:\, A';\, \cdots;\, A^{(s)}}(z_1, \cdots, z_s)t^n \tag{4.6}$$

$$= F_{\sigma:\, B';\, \cdots;\, B^{(s)};\, 0;\, 0}^{\nu:\, A';\, \cdots;\, A^{(s)};\, 1;\, 1}\left(\begin{matrix}[(\alpha_\nu):\theta', \cdots, \theta^{(s)}, \lambda, \lambda]: [(a'):\phi']; \cdots; \\ [(\beta_\sigma):\psi', \cdots, \psi^{(s)}, \mu, \mu]: [(b'):\delta']; \cdots;\end{matrix}\right.$$

$$\left.\begin{matrix}[(a^{(s)}):\phi^{(s)}]; [-\alpha:1]; [-\beta:1]; \\ [(b^{(s)}):\delta^{(s)}]: \quad ---; \quad ---;\end{matrix}\ z_1, \cdots, z_s, -\tfrac{1}{2}(x+1)t, -\tfrac{1}{2}(x-1)t\right),$$

whose special case when $\nu=\sigma=0$ and $z_1=\cdots=z_s=0$ yields the familiar result (see, for instance, [8], p. 593)

$$\sum_{n=0}^{\infty} P_n^{(\alpha-n,\beta-n)}(x)t^n=[1+\tfrac{1}{2}(x+1)t]^\alpha[1+\tfrac{1}{2}(x-1)t]^\beta . \tag{4.7}$$

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